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Vertex operators for quantum groups and application to integrable systems

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Abstract
Starting with any R-matrix with spectral parameters, and obeying the Yang–Baxter equation and a unitarity condition, we construct the corresponding infinite-dimensional quantum group $\mathcal{U}_R$ in terms of a deformed oscillator algebra $\mathcal{A}_R$. The realization we present is an infinite series, very similar to a vertex operator. Then, considering the integrable hierarchy naturally associated with $\mathcal{A}_R$, we show that $\mathcal{U}_R$ provides its integrals of motion. The construction can be applied to any infinite-dimensional quantum group, e.g. Yangians or elliptic quantum groups. Taking as an example the R-matrix of $Y(N)$, the Yangian based on $gl(N)$, using this construction we recover the nonlinear Schrödinger equation and its $Y(N)$ symmetry.

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1. Introduction

In this paper, our aim is to present a general construction of infinite-dimensional quantum groups as explicit integrals of motions of integrable systems. The construction relies only on the existence of an evaluated R-matrix (with spectral parameters) which obeys the unitarity condition. Thus, it can be applied to any infinite-dimensional quantum group.

To the R-matrix, one can associate a Zamolodchikov–Faddeev (ZF) algebra $\mathcal{A}_R$ [1], which in a Fock space representation provides the asymptotic states of the model. The quantum group is then constructed as an infinite series in the ZF generators, and is shown to commute with the Hamiltonian of the hierarchy. Thus, it generates the integrals of motion of the hierarchy. Moreover, since there is a natural action of the quantum group on the $\mathcal{A}_R$ generators, its action on the asymptotic states of the system is easily deduced.

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Taking as an example the R-matrix of $Y(N)$, the Yangian based on $gl(N)$, using this construction we recover the nonlinear Schrödinger (NLS) equation and its $Y(N)$ symmetry [2, 3]. It is thus very natural to believe that the other integrable systems found in the literature can be treated with the present approach.

The paper is organized as follows. In section 2, we introduce the different definitions and properties that are needed. From these notions, we construct, in section 3, a quantum group $\mathcal{U}_R$ from the deformed oscillator algebra $\mathcal{A}_R$. We consider in section 4 the hierarchy associated with $\mathcal{A}_R$ and show that $\mathcal{U}_R$ generates integrals of motion. Then, its Fock space representation is studied in section 5. Section 6 deals with three examples: the NLS equation with its Yangian symmetry (in the case of an R-matrix with additional spectral parameters), and $\mathcal{U}_q(\widehat{gl_2})$ and $\mathcal{A}_{q,p}(gl_2)$ (in the case of a multiplicative R-matrix). Finally, we conclude in section 7.

2. Definitions and first properties

2.1. ZF algebra

We start with an R-matrix satisfying the Yang–Baxter equation with spectral parameters

$$R_{12}(k_1, k_2)R_{13}(k_1, k_3)R_{23}(k_2, k_3) = R_{23}(k_2, k_3)R_{13}(k_1, k_3)R_{12}(k_1, k_2) \qquad (2.1)$$

and the unitarity condition

$$R_{12}(k_1, k_2)R_{21}(k_2, k_1) = \mathbb{I} \otimes \mathbb{I}. \qquad (2.2)$$

R is an $N^2 \times N^2$ matrix. Here and below, for brevity we denote

$$R_{12} \equiv R_{12}(k_1, k_2) \qquad (2.3)$$

but let us stress that the R-matrix we consider is defined with a spectral parameter. Note also that both the usual additive and multiplicative cases for the R-matrix, where $R(k_1, k_2)$ represents $R(k_1 - k_2)$ and $R(k_1/k_2)$ respectively, are included in our formalism.

Definition 2.1 (ZF algebra $\mathcal{A}_R$)**.** *To each R-matrix obeying equations (2.1) and (2.2), one can associate a ZF algebra $\mathcal{A}_R$ [1], with generators $a_i(k)$ and $a_i^\dagger(k)$ $(i = 1, \ldots, N)$ and exchange relations*

$$a_1 a_2 = R_{21} a_2 a_1 \qquad (2.4)$$

$$a_1^\dagger a_2^\dagger = a_2^\dagger a_1^\dagger R_{21} \qquad (2.5)$$

$$a_1 a_2^\dagger = a_2^\dagger R_{12} a_1 + \delta_{12}. \qquad (2.6)$$

We have used the notations

$$a_1 = \sum_{i=1}^{N} a_i(k_1) e_i \otimes \mathbb{I} \qquad a_2 = \sum_{i=1}^{N} a_i(k_2) \mathbb{I} \otimes e_i$$

$$a_1^\dagger = \sum_{i=1}^{N} a_i^\dagger(k_1) e_i^\dagger \otimes \mathbb{I} \qquad a_2^\dagger = \sum_{i=1}^{N} a_i^\dagger(k_2) \mathbb{I} \otimes e_i^\dagger$$

$$\delta_{12} = \delta(k_1 - k_2) \sum_{i=1}^{N} e_i \otimes e_i^\dagger \qquad e_i^\dagger = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \qquad e_i^\dagger \cdot e_j = \delta_{ij}$$

where $\cdot$ represents the vector scalar product.

Let us remark that, in the same way that the Yang–Baxter equation ensures the associativity of the product in $\mathcal{A}_R$, the unitarity condition can be interpreted as a consistency condition for the $\mathcal{A}_R$ algebra. Indeed, starting with equation (2.4), exchanging the auxiliary spaces $1 \leftrightarrow 2$ and the spectral parameters $k_1 \leftrightarrow k_2$, and multiplying by $(R_{12})^{-1}$, one obtains

$$a_1 a_2 = (R_{12})^{-1} a_2 a_1. \tag{2.7}$$

Comparing this last relation with equation (2.4), we recover the unitarity condition.

Above and in the following, we loosely write $a_1 \in \mathcal{A}_R$.

Property 2.2 (Adjoint anti-automorphism)**.**

$$\textit{Let } \dagger \textit{ be the operation defined by } \begin{cases} \mathcal{A}_R & \rightarrow \quad \mathcal{A}_R \\ a(k) & \mapsto \quad a^\dagger(k) \\ a^\dagger(k) & \mapsto \quad a(k) \\ R_{12}(k_1, k_2) & \mapsto \quad R_{21}(k_2, k_1) \end{cases} \tag{2.8}$$

and $(xy)^\dagger = y^\dagger x^\dagger \, \forall x, y \in \mathcal{A}_R$. *Then* $\dagger$ *is an automorphism of the* $\mathcal{A}_R$ *algebra, and we can identify* $(a)^\dagger \equiv a^\dagger$ *and* $(a^\dagger)^\dagger \equiv a$.

Proof. Direct calculation. For instance

$$(a_1 a_2)^\dagger = (a_2)^\dagger (a_1)^\dagger = (a_1)^\dagger (a_2)^\dagger (R_{21})^\dagger = (a_1)^\dagger (a_2)^\dagger R_{12}.$$

After the exchange $1 \leftrightarrow 2$, one recovers equation (2.5)

$$(a_1)^\dagger (a_2)^\dagger = (a_2)^\dagger (a_1)^\dagger R_{21}. \tag{2.9}$$

The other relations are obtained in the same way, once one remarks $(\delta_{21})^\dagger = \delta_{12}$. □

2.2. Vertex operators

Definition 2.3 (Vertex operators)**.** *The vertex operators* $T^{ij}(k)$ $(i, j = 1, \ldots, N)$ *associated with the algebra* $\mathcal{A}_R$ *are defined by* $T(k) \equiv T^{ij}(k) E_{ij} \in \mathcal{A}_R \otimes \mathbb{C}^{N^2}$ *where*

$$T(k_\infty) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a^\dagger_{n\ldots 1} T^{(n)}_{\infty 1\ldots n} a_{1\ldots n} \tag{2.10}$$

with

$$a^\dagger_{n\ldots 1} = a^\dagger_{\alpha_n}(k_n) \cdots a^\dagger_{\alpha_1}(k_1) \tag{2.11}$$

$$a_{1\ldots n} = a_{\beta_1}(k_1) \cdots a_{\beta_n}(k_n) \tag{2.12}$$

$$T^{(n)}_{\infty 1\ldots n} = T^{(n)}_{\infty, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n}(k_\infty, k_1, \ldots, k_n) \in \left(\mathbb{C}^{\otimes N^2}\right)^{\otimes(n+1)}(k_\infty, k_1, \ldots, k_n). \tag{2.13}$$

In equation (2.10), there is an implicit summation on the indices $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n = 1, \ldots, N$ *and an integration over the spectral parameters* $k_1, \ldots, k_n$.

For convenience, ∞ *denotes the auxiliary space associated with* $T(k_\infty)$*, and, as for the R-matrix, we note that* $T_\infty \equiv T_\infty(k_\infty)$.

Let us stress that, in the notation (2.10), the auxiliary spaces $1, \ldots, n$ are 'internal' in the sense that the indices corresponding to these spaces are summed and define scalars, not matrices, in these spaces. It is only the indices corresponding to the 'external' auxiliary space ∞ which refers to the matrix labelling for T. For instance $a_1^\dagger T^{(1)}_{\infty 1} a_1$ represents

$$a_1^\dagger T^{(1)}_{\infty 1} a_1 = \sum_{\alpha,\beta=1}^{N} \left(a_1^\dagger T^{(1)}_{\infty 1} a_1\right)_{\alpha,\beta} E_{\alpha,\beta} = \sum_{\alpha,\beta=1}^{N} \left(\sum_{\gamma,\mu=1}^{N} a^\dagger_\gamma T^{(1)}_{\alpha,\beta;\gamma,\mu} a_\mu \right) E_{\alpha,\beta}$$

so that we could also have written $a_2^\dagger T^{(1)}_{\infty 2} a_2$; $1, \ldots, n$ are dummy space indices.

Remark 1. The series (2.10) is very similar to a normal ordered (in a and $a^\dagger$) exponential

$$V(k_\infty) =: \exp(-a^\dagger M a): \tag{2.14}$$

whence the denomination vertex operator is used here to denote it.

Property 2.4 ($\mathcal{S}_n$-covariance of the vertex operators)**.** *The vertex operator coefficients $T^{(n)}_{\infty 1\ldots n}$ are covariant under the action of the permutation group $\mathcal{S}_n$.*

More precisely, for $\sigma \in \mathcal{S}_n$, one has

$$T^{(n)}_{\infty\sigma(1)\ldots\sigma(n)} = \mathcal{R}^{1\ldots n}_\sigma T^{(n)}_{\infty 1\ldots n} \left(\mathcal{R}^{1\ldots n}_\sigma\right)^{-1} \tag{2.15}$$

where $\mathcal{R}^{1\ldots n}_\sigma$ is the product of R-matrices defined by $a_{\sigma(1)\ldots\sigma(n)} = \mathcal{R}^{1\ldots n}_\sigma a_{1\ldots n}$.

Proof. Starting from the term $X_n = a^\dagger_{n\ldots 1} T^{(n)}_{\infty 1\ldots n} a_{1\ldots n}$ and relabelling the auxiliary spaces $i \to \sigma(i)$ (and also the spectral parameters), one obtains

$$X_n = a^\dagger_{\sigma(n)\ldots\sigma(1)} T^{(n)}_{\infty\sigma(1)\ldots\sigma(n)} a_{\sigma(1)\ldots\sigma(n)}. \tag{2.16}$$

Then, from the exchange properties of a and $a^\dagger$ and the property 2.2, one has

$$a^\dagger_{\sigma(n)\ldots\sigma(1)} = a^\dagger_{n\ldots 1}\left(\mathcal{R}^{1\ldots n}_\sigma\right)^{-1} \qquad \text{and} \qquad a_{\sigma(1)\ldots\sigma(n)} = \mathcal{R}^{1\ldots n}_\sigma a_{1\ldots n} \tag{2.17}$$

which leads to the formula (2.15). □

As an example, if σ is just the transposition $i \leftrightarrow i+1$, one obtains $\mathcal{R}^{1\ldots n}_\sigma = R_{i,i+1}$ and the formula

$$T^{(n)}_{\infty 1\ldots i-1,i+1,i,i+2\ldots n} = R_{i,i+1} T^{(n)}_{\infty 1\ldots n} R_{i+1,i}. \tag{2.18}$$

Property 2.5. *The matrices $\mathcal{R}^{1\ldots n}_\sigma$, $\sigma \in \mathcal{S}_n$, defined by*

$$a_{\sigma(1)\ldots\sigma(n)} = \mathcal{R}^{1\ldots n}_\sigma a_{1\ldots n} \tag{2.19}$$

obey

$$\mathcal{R}^{\mu(1)\ldots\mu(n)}_\sigma \mathcal{R}^{1\ldots n}_\mu = \mathcal{R}^{1\ldots n}_{\sigma\circ\mu} \qquad \textit{so that} \qquad \left(\mathcal{R}^{1\ldots n}_\sigma\right)^{-1} = \mathcal{R}^{\sigma(1)\ldots\sigma(n)}_{\sigma^{-1}}. \tag{2.20}$$

From any matrix $M_{1\ldots n} \in \left(\mathbb{C}^{N^2}\right)^{\otimes n}$, one can construct a $\mathcal{S}_n$-covariant matrix by

$$\widetilde{M}_{1\ldots n} = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \left(\mathcal{R}^{1\ldots n}_\sigma\right)^{-1} M_{\sigma(1)\ldots\sigma(n)} \mathcal{R}^{1\ldots n}_\sigma. \tag{2.21}$$

Proof. The first formula is proven by direct calculation:

$$a_{\sigma\circ\mu(1)\ldots\sigma\circ\mu(n)} = \mathcal{R}^{1\ldots n}_{\sigma\circ\mu} a_{1\ldots n} = \mathcal{R}^{\mu(1)\ldots\mu(n)}_\sigma a_{\mu(1)\ldots\mu(n)} = \mathcal{R}^{\mu(1)\ldots\mu(n)}_\sigma \mathcal{R}^{1\ldots n}_\mu a_{1\ldots n}. \tag{2.22}$$

Now, for the last formula, one has (for any $\mu \in \mathcal{S}_n$)

$$\begin{aligned}\widetilde{M}_{\mu(1)\ldots\mu(n)} &= \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \left(\mathcal{R}^{\mu(1)\ldots\mu(n)}_\sigma\right)^{-1} M_{\sigma\circ\mu(1)\ldots\sigma\circ\mu(n)} \mathcal{R}^{\mu(1)\ldots\mu(n)}_\sigma \\ &= \frac{1}{n!}\sum_{\sigma'\in\mathcal{S}_n} \left(\mathcal{R}^{\mu(1)\ldots\mu(n)}_{\sigma'\circ\mu^{-1}}\right)^{-1} M_{\sigma'(1)\ldots\sigma'(n)} \mathcal{R}^{\mu(1)\ldots\mu(n)}_{\sigma'\circ\mu^{-1}}\end{aligned}$$

where in the last expression, we have made the change of variable $\sigma' = \sigma\circ\mu$. Now, using equation (2.20), one obtains $\mathcal{R}^{\mu(1)\ldots\mu(n)}_{\sigma'\circ\mu^{-1}} = \mathcal{R}^{1\ldots n}_{\sigma'}\left(\mathcal{R}^{1\ldots n}_\mu\right)^{-1}$, and $\left(\mathcal{R}^{\mu(1)\ldots\mu(n)}_{\sigma'\circ\mu^{-1}}\right)^{-1} = \mathcal{R}^{1\ldots n}_\mu\left(\mathcal{R}^{1\ldots n}_{\sigma'}\right)^{-1}$, so that $\widetilde{M}_{1\ldots n}$ is $\mathcal{S}_n$-covariant. □

Remark 2. Strictly speaking, one can start with vertex operators which do not obey the $\mathcal{S}_n$-covariance (2.15), but the relevant part in the vertex operator will be the covariant one, as given by equation (2.21).

2.3. Well-bred operators

Definition 2.6 (well-bred operators). *An operator L is said to be well-bred*[2] *(on $\mathcal{A}_R$) when it acts on a and $a^\dagger$ as*

$$L_1a_2 = R_{21}a_2L_1 \qquad \text{and} \qquad L_1a_2^\dagger = a_2^\dagger R_{12}L_1. \tag{2.23}$$

We give a few properties of well-bred operators that will be useful in the following.

Lemma 2.7. *Let L be a well-bred operator, then $L^\dagger(k)L(k)$ is central in $\mathcal{A}_R$.*

Proof. One applies the † automorphism to the relations (2.23). We obtain

$$a_2^\dagger L_1^\dagger = L_1^\dagger a_2^\dagger R_{12} \qquad \text{and} \qquad a_2L_1^\dagger = L_1^\dagger R_{21}a_2^\dagger. \tag{2.24}$$

Then a direct calculation shows that $L^\dagger(k)L(k)$ commutes with a and $a^\dagger$. For instance

$$L_1^\dagger L_1a_2^\dagger = L_1^\dagger a_2^\dagger R_{12}L_1 = a_2^\dagger L_1^\dagger L_1. \qquad \square$$

Lemma 2.8. *Let L be a well-bred operator of $\mathcal{A}_R$. Then $c_{12} = L_1^{-1}L_2^{-1}R_{12}L_1L_2$ is central in $\mathcal{A}_R$. It satisfies $c_{12}^{-1} = c_{21}$.*

Proof. Starting with equation (2.23), one obtains

$$L_1L_2a_3 = R_{32}R_{31}a_3L_1L_2 \tag{2.25}$$

which can be rewritten (after the exchange $1 \leftrightarrow 2$) as

$$R_{31}R_{32}a_3 = L_2L_1a_3L_1^{-1}L_2^{-1}. \tag{2.26}$$

Then

$$\begin{aligned} R_{12}L_1L_2a_3 &= R_{12}R_{32}R_{31}a_3L_1L_2 = R_{31}R_{32}R_{12}a_3L_1L_2 = R_{31}R_{32}a_3R_{12}L_1L_2 \\ &= L_2L_1a_3L_1^{-1}L_2^{-1}R_{12}L_1L_2 \end{aligned} \tag{2.27}$$

so that, multiplying by $L_1^{-1}L_2^{-1}$, we obtain

$$c_{12}a_3 = a_3c_{12}. \tag{2.28}$$

Performing a similar calculation with $a_3^\dagger$, we obtain $c_{12}a_3^\dagger = a_3^\dagger c_{12}$.

The last equation is a direct consequence of the unitarity condition. $\square$

3. Construction of well-bred vertex operators

We first give a characterization of well-bred vertex operators.

Lemma 3.1. *The vertex operator T is well-bred if and only if $T^{(n)}_{\infty 1\ldots n}$ obeys*

$$T^{(1)}_{\infty 0} = \mathbb{I} - R_{\infty 0} \qquad \text{and for} \qquad n \geqslant 1: \tag{3.1}$$

$$(n+1)\left\{T^{(n)}_{\infty 1\ldots n} - (\mathcal{R}_{0,n})^{-1}R_{\infty 0}T^{(n)}_{\infty 1\ldots n}\mathcal{R}_{0,n}\right\} = \sum_{i=1}^{n+1}(\mathcal{R}_{0,i-1})^{-1}T^{(n+1)}_{\infty 1\ldots n|i}\mathcal{R}_{0,i-1}$$

where we have introduced

$$\mathcal{R}_{0,n} = \overleftarrow{\prod_{a=1}^{n}} R_{0a};\ T^{(n+1)}_{\infty 1\ldots n|i} = T^{(n+1)}_{\infty 1\ldots i-1,0,i\ldots n}\,(i \leqslant n) \qquad \text{and} \qquad T^{(n+1)}_{\infty 1\ldots n|n+1} = T^{(n+1)}_{\infty 1\ldots n0}.$$

[2] We call these operators 'well-bred' because they act nicely (on a and $a^\dagger$).

Proof. We prove the property by a direct calculation. We note $\widehat{T}_\infty = T_\infty - \mathbb{I}$:

$$a_0\widehat{T}_\infty = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(a_n^\dagger R_{0n} a_0 + \delta_{0n}\right) a_{n-1\ldots 1}^\dagger T_{\infty 1\ldots n}^{(n)} a_{1\ldots n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left\{ a_{n\ldots 1}^\dagger R_{0n} \cdots R_{01} a_0 T_{\infty 1\ldots n}^{(n)} a_{1\ldots n} + \sum_{i=1}^{n} a_{n\ldots i+1}^\dagger a_{i-1\ldots 1}^\dagger R_{0n} \cdots R_{0i+1} \delta_{0i} T_{\infty 1\ldots n}^{(n)} a_{1\ldots n} \right\}.$$

Using

$$\delta_{0i} T_{\infty 1\ldots n}^{(n)} a_i = T_{\infty 1\ldots i-1,0,i+1\ldots n}^{(n)} a_0 \tag{3.2}$$

and after a relabelling $j \to j-1$ for $j \geqslant i+1$, one obtains

$$a_0\widehat{T}_\infty = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a_{n\ldots 1}^\dagger R_{0n} \cdots R_{01} a_0 T_{\infty 1\ldots n}^{(n)} a_{1\ldots n} - T_{\infty 0}^{(1)} a_0 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \sum_{i=1}^{n} a_{n-1\ldots 1}^\dagger R_{0n-1} \cdots R_{0i} T_{\infty 1\ldots 0i\ldots n-1}^{(n)} a_{1\ldots i-1} a_0 a_{i\ldots n-1}$$

with

$$i = n : R_{0n-1} \cdots R_{0i} \equiv 1,\ T_{\infty 1\ldots 0i\ldots n-1}^{(n)} \equiv T_{\infty 1\ldots n-1,0}^{(n)} \quad \text{and} \quad a_{1\ldots i-1} a_0 a_{i\ldots n-1} \equiv a_{1\ldots n-1} a_0$$

as a notation. Rewriting

$$a_{1\ldots i-1} a_0 a_{i\ldots n-1} = R_{0i-1} \cdots R_{01} a_0 a_{1\ldots n-1} \tag{3.3}$$

and relabelling $n \to n-1$ in the second summation we are led to

$$a_0\widehat{T}_\infty = -T_{\infty 0}^{(1)} a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left\{ a_{n\ldots 1}^\dagger R_{0n} \cdots R_{01} a_0 T_{\infty 1\ldots n}^{(n)} a_{1\ldots n} - \frac{1}{n+1} \sum_{i=1}^{n+1} a_{n\ldots 1}^\dagger R_{0n} \cdots R_{0i} T_{\infty 1\ldots n|i}^{(n+1)} R_{0i-1} \cdots R_{01} a_0 a_{1\ldots n} \right\} \tag{3.4}$$

that is

$$a_0\widehat{T}_\infty = -T_{\infty 0}^{(1)} a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a_{n\ldots 1}^\dagger \left\{ \mathcal{R}_{0n} T_{\infty 1\ldots n}^{(n)} - \frac{1}{n+1} \sum_{i=1}^{n+1} \mathcal{R}_{0n} \mathcal{R}_{0,i-1}^{-1} T_{\infty 1\ldots n|i}^{(n+1)} \mathcal{R}_{0,i-1} \right\} a_0 a_{1\ldots n}. \tag{3.5}$$

On the other hand, one computes

$$R_{\infty 0} \widehat{T}_\infty a_0 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a_{n\ldots 1}^\dagger R_{\infty 0} T_{\infty 1\ldots n}^{(n)} \mathcal{R}_{0n} a_0 a_{1\ldots n}. \tag{3.6}$$

Finally, making equations (3.5) and (3.6) equal, we obtain equations (3.1), after left-multiplication by $\mathcal{R}_{0n}^{-1}$.

A similar calculation on $T_\infty a_0^\dagger = a_0^\dagger R_{\infty 0} T_\infty$ leads to the same equation. □

Remark 3. If one defines $\mathcal{R}_{00} = \mathbb{I}$ (and $T_\infty^{(0)} = \mathbb{I}$ as given by equation (2.10)), equation $T_{\infty 0}^{(1)} = \mathbb{I} - R_{\infty 0}$ just corresponds to $n = 0$ in equation (3.1).

Property 3.2 (Central generators of $\mathcal{A}_R$). *The only central generators of $\mathcal{A}_R$ are constants.*

Proof. Let c be a central generator of $\mathcal{A}_R$. Since it commutes with a and $a^\dagger$, it also commutes with the number operator $H_0 = \int \mathrm{d}k\, a^\dagger(k)a(k)$ (see section 4). It is thus of the form

$$c = c^{(0)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a^\dagger_{n\ldots 1} c^{(n)}_{1\ldots n} a_{1\ldots n}. \tag{3.7}$$

Demanding $ca_0 = a_0 c$ leads to equations on the elements $c^{(n)}_{1\ldots n}$. These equations are computed in the same way one computes the equations for $T^{(n)}$. Indeed, one deduces the equations on $c^{(n)}$ by formally replacing $R_{0\infty}$ by $\mathbb{I}$ in equations (3.1). We obtain the relations

$$c^{(1)}_1 = 0 \qquad (n+1)\left\{c^{(n)}_{1\ldots n} - (\mathcal{R}_{0,n})^{-1} c^{(n)}_{1\ldots n} \mathcal{R}_{0,n}\right\} = \sum_{i=1}^{n+1} \mathcal{R}^{-1}_{0,i-1} c^{(n+1)}_{1\ldots n|i} \mathcal{R}_{0,i-1} \qquad \text{for} \quad n \geqslant 1. \tag{3.8}$$

We prove by induction that $c^{(n)} = 0$. The case $n = 1$ is a direct consequence of the equations. Let us suppose that $c^{(p)} = 0$ for $p \leqslant n$. Writing the equation (3.8) at level n, and using the induction, we have

$$\sum_{i=1}^{n+1} \mathcal{R}^{-1}_{0,i-1} c^{(n+1)}_{1\ldots n|i} \mathcal{R}_{0,i-1} = 0. \tag{3.9}$$

Using the invariance property 2.4, we can rewrite each term of the sum as

$$c^{(n+1)}_{1\ldots n|i} = c^{(n+1)}_{1\ldots i-1,0,i\ldots n} = \mathcal{R}_{0,i-1} c^{(n+1)}_{01\ldots n} \mathcal{R}^{-1}_{0,i-1}. \tag{3.10}$$

Thus, the equation is equivalent to $(n+1)c^{(n+1)}_{01\ldots n} = 0$ and the induction is proven. □

Theorem 3.3. *The vertex operator T is well-bred if and only if $T^{(n)}_{\infty 1\ldots n}$ is defined by the following inductive expressions:*

$$T^{(1)}_{\infty 0} = \mathbb{I} - R_{\infty 0} \tag{3.11}$$

$$T^{(n+1)}_{\infty 01\ldots n} = \frac{1}{n+1} \sum_{i=0}^{n} \left(\mathcal{R}^{01\ldots n}_{p_i}\right)^{-1} T^{(n)}_{\infty 2\ldots i,0,i+1,\ldots,n} \mathcal{R}^{01\ldots n}_{p_i} - \frac{1}{(n+1)!} \sum_{\sigma \in \mathcal{S}_{n+1}} \left(\mathcal{R}^{01\ldots n}_{p_n \circ \sigma}\right)^{-1} R_{\infty \sigma(0)} T^{(n)}_{\infty \sigma(1)\ldots\sigma(n)} \mathcal{R}^{01\ldots n}_{p_n \circ \sigma} \tag{3.12}$$

where $T^{(n)}_{\infty 2\ldots i,0,i+1,\ldots,n}$ for $i = 0$ stands for $T^{(n)}_{\infty 1\ldots n}$. $p_j \in \mathcal{S}_{n+1}$ is defined by

$$p_j : (0, 1, \ldots, j-1, j, j+1, \ldots, n) \to (1, 2, \ldots, j, 0, j+1, \ldots, n),\ 1 \leqslant j \leqslant n$$
$$p_0 = id. \tag{3.13}$$

We note that R_{ij} represents $R_{ij}(k_i, k_j)$.

Proof. We start with lemma 3.1 and show that T obeys the above inductive expressions. We remark that, from the definition of $\mathcal{R}_{0,i}$, one has

$$\mathcal{R}_{0,i} a_{01\ldots n} = a_{1,2,\ldots,i,0,i+1\ldots n} \Rightarrow \mathcal{R}_{0,i} = \mathcal{R}^{01\ldots n}_{p_i} \tag{3.14}$$

where p_i is defined by equation (3.13). We start from equation (3.1) and work with $\mathcal{S}$-covariant matrices. Then, the right-hand side (rhs) is equal to $(n+1)T^{(n+1)}_{\infty 01\ldots n}$, while the left-hand side reads

$$\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_{n+1}} \left(\mathcal{R}^{01\ldots n}_{\sigma}\right)^{-1} \left\{T^{(n)}_{\infty\sigma(1)\ldots\sigma(n)} - \left(\mathcal{R}^{\sigma(0)\ldots\sigma(n)}_{p_n}\right)^{-1} R_{\infty\sigma(0)} T^{(n)}_{\infty\sigma(1)\ldots\sigma(n)} \mathcal{R}^{\sigma(0)\ldots\sigma(n)}_{p_n}\right\} \mathcal{R}^{01\ldots n}_{\sigma}. \tag{3.15}$$

Now, we decompose $\mathcal{S}_{n+1}$ with respect to $\mathcal{S}_n$; any $\sigma \in \mathcal{S}_{n+1}$ is of the form (for some $0 \leqslant i \leqslant n$) $\mu \circ p_i$ with[3] $\mu \in \mathcal{S}_n$ and p_i defined in equation (3.13). Using the covariance of $T^{(n)}$, one obtains for the first part of the rhs

$$\begin{aligned}
\text{rhs}_1 := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_{n+1}} \left(\mathcal{R}_\sigma^{01\ldots n}\right)^{-1} T^{(n)}_{\infty\sigma(1)\ldots\sigma(n)} \mathcal{R}_\sigma^{01\ldots n} \\
&= \frac{1}{n!} \sum_{i=0}^{n} \sum_{\mu \in \mathcal{S}_n} \left(\mathcal{R}_{p_i}^{01\ldots n}\right)^{-1} \left(\mathcal{R}_\mu^{p_i(0)p_i(1)\ldots p_i(n)}\right)^{-1} T^{(n)}_{\infty\mu(p_i(1))\ldots\mu(p_i(n))} \\
&\quad \times \mathcal{R}_\mu^{p_i(0)p_i(1)\ldots p_i(n)} \mathcal{R}_{p_i}^{01\ldots n} \\
&= \frac{1}{n!} \sum_{i=0}^{n} \left(\mathcal{R}_{p_i}^{01\ldots n}\right)^{-1} \left(\sum_{\mu \in \mathcal{S}_n} \mathcal{T}^{(\mu)}_{\infty p_i(0)p_i(1)\ldots p_i(n)} \right) \mathcal{R}_{p_i}^{01\ldots n}
\end{aligned}$$

with $\mathcal{T}^{(\mu)}_{\infty 01\ldots n} = \left(\mathcal{R}_\mu^{01\ldots n}\right)^{-1} T^{(n)}_{\infty\mu(1)\ldots\mu(n)} \mathcal{R}_\mu^{01\ldots n}$. Since $\mu(0) = 0$, one has $\mathcal{R}_\mu^{01\ldots n} = \mathcal{R}_\mu^{1\ldots n}$. Then, using the $\mathcal{S}_n$-covariance of $T^{(n)}$, one obtains $\mathcal{T}^{(\mu)}_{\infty 01\ldots n} = T^{(n)}_{\infty 1\ldots n}$, $\forall \mu$, so that

$$\text{rhs}_1 = \sum_{i=0}^{n} \left(\mathcal{R}_{p_i}^{01\ldots n}\right)^{-1} T^{(n)}_{\infty 2\ldots i,0,i+1\ldots n} \mathcal{R}_{p_i}^{01\ldots n}. \tag{3.16}$$

Finally, to obtain equation (3.12), one remarks in the second sum of the rhs that $\mathcal{R}_{p_n}^{\sigma(0)\ldots\sigma(n)} \mathcal{R}_\sigma^{01\ldots n} = \mathcal{R}_{p_n \circ \sigma}^{01\ldots n}$, due to equation (2.20).

The same calculation (done in the reverse direction) also shows that the inductive expressions obey lemma 3.1. □

Remark 4. Note that the inductive expression proves the unicity of the solution.

Remark 5. The first terms in the series (3.12) are

$$\begin{aligned}
T^{(1)}_{\infty 1} &= \mathbb{I} - R_{\infty 1} \\
T^{(2)}_{\infty 12} &= \mathbb{I} - R_{\infty 2} + R_{\infty 2} R_{\infty 1} - R_{21} R_{\infty 1} R_{12}.
\end{aligned}$$

Corollary 3.4. *$\forall n \geqslant 0$, $T^{(n)}_{\infty 1\ldots n}$ is a non-vanishing polynomial of R-matrices. It has the following form*

$$T^{(n)}_{\infty 1\ldots n} = \mathbb{I} + \sum_{i=1}^{n} S^{(i)}_{\infty 1\ldots n} \qquad \text{with} \quad S^{(i)}_{\infty 1\ldots n} = \sum_{\mu \in \mathcal{S}_n} m_\mu M_\mu R_{\infty\mu(1)} \cdots R_{\infty\mu(i)} M_\mu^{-1} \tag{3.17}$$

where M_μ are products of matrices R_{ab} with $1 \leqslant a, b \leqslant n$ and $m_\mu \in \mathbb{Z}$.

Proof. We prove the corollary by induction. The explicit expressions given above prove that it is true for $n = 0, 1, 2$. Now, suppose equation (3.17) is true up to n. Then, equation (3.12) shows that it is also true for $n+1$. Indeed, the two sums in equation (3.12) have conjugation by R-matrices of type M_μ. Moreover, only the first sum contributes to $\mathbb{I}$, and effectively leads to a coefficient 1, while the second sum increases the number of $R_{\infty a}$ ($a = 0, 1, \ldots, n$) matrices by 1. □

Remark 6. The above formula shows that $T^{(n)}$ is invertible (as a series) for all n.

Using the theorem 3.3, one can show the following property.

[3] Strictly speaking, μ is still in $\mathcal{S}_{n+1}$, but it obeys $\mu(0) = 0$ so that its restriction to $[1, n]$ defines an element of $\mathcal{S}_n$.

Property 3.5. *The well-bred vertex operators T of theorem 3.3 obey Faddeev–Reshetikhin–Takhtajan (FRT) relations:*

$$R_{12}T_1T_2 = T_2T_1R_{12} \qquad \text{i.e.} \quad R_{12}(k_1,k_2)T_1(k_1)T_2(k_2) = T_2(k_2)T_1(k_1)R_{12}(k_1,k_2). \tag{3.18}$$

In other words, they generate an infinite-dimensional quantum group with the evaluated R-matrix R_{12}. In the following, we denote this quantum group as $\mathcal{U}_R$.

Proof. We use lemma 2.8 for T. $c_{12} = T_1^{-1}T_2^{-1}R_{12}T_1T_2$ is central in $\mathcal{A}_R$ such that

$$R_{12}T_1T_2 = T_2T_1c_{12} \tag{3.19}$$

with c_{12} being central, and due to the property 3.2, it is a constant matrix M_{12}. To identify the exact expression of M_{12}, we use the result of theorem 3.3. Looking at equation (3.19) as a series in the number of say a operators and projecting on number 0, we obtain $c_{12} = M_{12} = R_{12}$. □

Remark 7. Looking at the term linear in a, one obtains

$$R_{12}\big(T_{13}^{(1)} + T_{23}^{(1)} - T_{13}^{(1)}\cdot T_{23}^{(1)}\big) = \big(T_{13}^{(1)} + T_{23}^{(1)} - T_{13}^{(1)}\cdot T_{13}^{(1)}\big)c_{12}. \tag{3.20}$$

Plugging into this equation the expressions of $T^{(1)}$ and c_{12}, one recovers the Yang–Baxter equation, which is indeed satisfied.

Property 3.6. *Let T be the well-bred vertex operator of theorem 3.3. Then, one has*

$$T^{\dagger}(k) = T(k)^{-1}. \tag{3.21}$$

Proof. From the lemma 2.7, one knows that $T^{\dagger}(k)T(k)$ is central. This implies (using property 3.2) that $T^{\dagger}(k)T(k)$ is a constant $N\times N$ matrix M. Looking at the term without a, one concludes that $M = \mathbb{I}_N$. □

Corollary 3.7. *The expansion of $T(k)^{-1}$ as a series in a takes the form*

$$T_{\infty}^{-1} = \mathbb{I} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}a^{\dagger}_{n\ldots 1}\bar{T}^{(n)}_{\infty 1\ldots n}a_{1\ldots n} \tag{3.22}$$

where $\bar{T}^{(n)}_{\infty 1\ldots n}$ is defined by the following inductive expressions

$$\bar{T}^{(1)}_{\infty 0} = \mathbb{I} - R_{0\infty} \tag{3.23}$$

$$\bar{T}^{(n+1)}_{\infty 01\ldots n} = \frac{1}{n+1}\sum_{i=0}^{n}\big(\mathcal{R}^{01\ldots n}_{p_i}\big)^{-1}\bar{T}^{(n)}_{\infty 2\ldots i,0,i+1,\ldots,n}\mathcal{R}^{01\ldots n}_{p_i}$$
$$-\frac{1}{(n+1)!}\sum_{\sigma\in\mathcal{S}_{n+1}}\big(\mathcal{R}^{01\ldots n}_{p_n\circ\sigma}\big)^{-1}\bar{T}^{(n)}_{\infty\sigma(1)\ldots\sigma(n)}R_{\sigma(0)\infty}\mathcal{R}^{01\ldots n}_{p_n\circ\sigma}. \tag{3.24}$$

It obeys the corollary 3.4, with $R_{\infty\mu(i)}$ replaced by $R_{\mu(i)\infty}$.

Proof. A simple calculation can be made from property 2.2, theorem 3.3 and property 3.6. □

Property 3.8. *The vertex operators T defined in theorem 3.3 induce an isomorphism between the algebras $\mathcal{A}_R$ and $\mathcal{A}_{R^{-1}}$. The isomorphism is given by*

$$\tau : \begin{cases} \mathcal{A}_R & \rightarrow \quad \mathcal{A}_{R^{-1}} \\ a & \mapsto \quad \hat{a} = T^{-1}a \\ a^\dagger & \mapsto \quad \hat{a}^\dagger = a^\dagger T. \end{cases} \tag{3.25}$$

Proof. We first show that $\hat{a}$ and $\hat{a}^\dagger$ obey the exchange relations of $\mathcal{A}_{R^{-1}}$. We note that $R_{12}^{-1} = R_{21}$.

$$\begin{aligned} \hat{a}_1\hat{a}_2 &= T_1^{-1}a_1T_2^{-1}a_2 = T_1^{-1}T_2^{-1}R_{12}a_1a_2 = R_{12}T_2^{-1}T_1^{-1}a_1a_2 \\ &= R_{12}T_2^{-1}T_1^{-1}R_{21}a_2a_1 = R_{12}T_2^{-1}a_2T_1^{-1}a_1 = R_{12}\hat{a}_2\hat{a}_1. \end{aligned}$$

One does a similar calculation with $\hat{a}_1^\dagger\hat{a}_2^\dagger$. In the same way, one computes

$$\begin{aligned} \hat{a}_1\hat{a}_2^\dagger &= T_1^{-1}a_1a_2^\dagger T_2 = T_1^{-1}a_2^\dagger R_{12}a_1T_2 + T_1^{-1}\delta_{12}T_2 = T_1^{-1}a_2^\dagger T_2a_1 + \delta_{12} \\ &= a_2^\dagger T_1^{-1}R_{21}T_2a_1 + \delta_{12} = a_2^\dagger T_2R_{21}T_1^{-1}a_1 + \delta_{12} = \hat{a}_2^\dagger R_{21}\hat{a}_1 + \delta_{12}. \end{aligned}$$

This shows that $\mathcal{A}_R$ is embedded into $\mathcal{A}_{R^{-1}}$. Performing the same calculation starting from $\mathcal{A}_{R^{-1}}$ proves that $\mathcal{A}_{R^{-1}}$ is embedded into $\mathcal{A}_R$. There is thus an equality of the two algebras. □

Reduction to the finite-dimensional case. The above results can be applied to the case without spectral parameters. We have to start with a finite-dimensional R-matrix obeying the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3.26}$$

and a unitarity condition $R_{12}R_{21} = \mathbb{I}$ where, for this section only, the spectral parameters are not present. The deformed oscillator algebra is then finite dimensional, and all the properties stated above are still valid, the proofs following the same lines, omitting the integration over the spectral parameters.

Note however that the unitarity condition still has to be fulfilled, and this requirement excludes for instance the (triangular) R-matrix of the finite-dimensional quantum group $\mathcal{U}_q(sl_2)$.

4. Application to integrable systems

Property 4.1 (Hierarchy associated with $\mathcal{A}_R$)**.** *Let $H^{(n)}$ be defined by*

$$H^{(n)} = \int_{-\infty}^{\infty} \mathrm{d}k\, k^n a^\dagger(k)a(k) \qquad \forall n = 0, 1, 2, \ldots. \tag{4.1}$$

$H^{(n)}$ forms an Abelian algebra, which defines a hierarchy for the algebra $\mathcal{A}_R$.

The evolution of a and $a^\dagger$ operators under the flow $H^{(n)}$ is given by

$$\mathrm{e}^{\mathrm{i}tH^{(n)}}a(k)\,\mathrm{e}^{-\mathrm{i}tH^{(n)}} = \mathrm{e}^{-\mathrm{i}tk^n}a(k) \tag{4.2}$$

$$\mathrm{e}^{\mathrm{i}tH^{(n)}}a^\dagger(k)\,\mathrm{e}^{-\mathrm{i}tH^{(n)}} = \mathrm{e}^{\mathrm{i}tk^n}a^\dagger(k). \tag{4.3}$$

Proof. Direct calculation. For instance

$$a_1H^{(n)} = a_1k_2^na_2^\dagger a_2 = k_2^n\left(a_2^\dagger R_{12}a_1a_2 + \delta_{12}a_2\right) = k_2^na_2^\dagger R_{12}R_{21}a_2a_1 + k_1^na_1 = H^{(n)}a_1 + k_1^na_1$$

and thus $[H^{(n)}, a_1] = -k_1^na_1$. □

Property 4.2. *Any well-bred operator L is an integral of motion for the hierarchy*

$$[L, H^{(n)}] = 0 \qquad \forall n = 0, 1, 2, \ldots. \tag{4.4}$$

In particular, this is the case for the well-bred vertex operators defined in theorem 3.3, and the quantum group $\mathcal{U}_R$ generates an infinite-dimensional symmetry algebra for the hierarchy.

Proof. $L_1 H_2^{(m)} = k_2^m L_1 a_2^\dagger a_2 = k_2^m a_2^\dagger R_{12} L_1 a_2 = k_2^m a_2^\dagger R_{12} R_{21} a_2 L_1 = H_2^{(m)} L_1.$ □

Remark 8. From the example of section 6.1 (see below), which was studied in [4, 2], we conjecture that, for each $\mathcal{A}_R$-hierarchy, there is a corresponding integrable system already studied in the literature. The $a^\dagger$ operators in the Fock space representation, in this context, correspond to asymptotic states of the system. The correlation functions of the system are then computed using the $a^\dagger$ operators.

5. Fock space and evaluation representations

Associated with the deformed oscillator algebra $\mathcal{A}_R$ comes the notion of Fock space.

Definition 5.1. *The Fock space $\mathcal{F}_R$ of the $\mathcal{A}_R$ algebra is the module generated by the vacuum Ω such that*

$$a_i(k)\Omega = 0 \qquad \forall i = 1, \ldots, N\ \forall k. \tag{5.1}$$

Now, since one has constructed a quantum group from the $\mathcal{A}_R$ algebra, it is natural to look at the representations induced by the Fock space.

Property 5.2. *The Fock space $\mathcal{F}_R$ decomposes under the action of the Hamiltonians $H^{(n)}$ into an infinite sum of tensor product of evaluation representations of $\mathcal{U}_R$*

$$\mathcal{F}_R = \oplus_{n=0}^{\infty} \int \mathrm{d}k_1 \cdots \mathrm{d}k_n \theta(k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n) \mathcal{V}_n(k_1, k_2, \ldots, k_n) \tag{5.2}$$

where $\theta(k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n)$ indicates that the spectral parameters are ordered.

In particular, the representations $\mathcal{V}_n(k_1, \ldots, k_n)$ are of dimension N^n, and T acts in these spaces by right-multiplication by R.

Proof. Since the Hamiltonians $H^{(n)}$ form a commuting subalgebra of $\mathcal{A}_R$, we can consider them as a Cartan subalgebra, and decompose $\mathcal{F}_R$ into Cartan eigenspaces $\mathcal{V}_n(h_1, h_2, \ldots)$, where n denotes the eigenvalue under $H^{(0)}$ (which turns out to be still the particle number although we are in the deformed case) and h_p is the eigenvalue of $H^{(p)}$ ($p > 0$). Now, since $\mathcal{U}_R$ commutes with these Hamiltonians, the eigenspaces are stable under the action of $\mathcal{U}_R$ and thus are representations of $\mathcal{U}_R$.

The vectors in $\mathcal{F}_R$ are linear combinations of monomials $a_{\alpha_1}^\dagger(k_1) \cdots a_{\alpha_m}^\dagger(k_m)\Omega$, $\forall m$. On the eigenspace $\mathcal{V}_n(h_0, h_1, h_2, \ldots)$, one must consider only monomials with $m = n$; this provides only a finite number of terms, and the eigenspace is of finite dimension. Moreover, the eigenvalues under $H^{(n)}$ being fixed, one has equations

$$h_1 = \sum_{i=1}^n k_i, h_2 = \sum_{i=1}^n k_i^2, \ldots, h_n = \sum_{i=1}^n k_i^n$$

which completely fix the values of $k_1, \ldots, k_n$ (up to a permutation) and also of $h_p = \sum_{i=1}^n k_i^p$, $p > n$. Thus, we can replace the labelling $h_1, h_2, \ldots$ by $k_1, \ldots, k_n$, whence the notation $\mathcal{V}_n(k_1, \ldots, k_n)$ for the representations of $\mathcal{U}_R$. Finally, the exchange relations among $a^\dagger$ allow us to reorder them in such a way that the spectral parameters are in increasing order.

Because it is a vertex operator, the action of T on Ω is trivial, and since it is well-bred its action on other states is a multiplication by R. □

Remark 9 (Hopf structure of $\mathcal{U}_R$)**.** Although one cannot obtain the Hopf structure of $\mathcal{U}_R$ starting from $\mathcal{A}_R$, one can infer it from the present construction in the following way.

The 'first' eigenspaces are

$$\mathcal{V}_0(0) = \mathbb{C}\Omega$$
$$\mathcal{V}_1(k) = \text{Span}\big(a_i^\dagger(k)\Omega, i = 1, \ldots, N\big)$$
$$\mathcal{V}_2(k_1, k_2) = \text{Span}\big(a_j^\dagger(k_2)a_i^\dagger(k_1)\Omega, k_1 \leqslant k_2, i, j = 1, \ldots, N\big).$$

Looking at the action of the well-bred vertex operators T on these spaces, one obtains

$$T\Omega = \Omega \qquad T_1 a_2^\dagger\Omega = a_2^\dagger R_{12}\Omega \qquad T_1 a_2^\dagger a_3^\dagger\Omega = a_2^\dagger R_{12} a_3^\dagger R_{13}\Omega. \tag{5.3}$$

Interpreting $\mathcal{V}_2(k_1, k_2)$ as the tensor product $\mathcal{V}_1(k_1) \otimes \mathcal{V}_1(k_2)$

$$a_2^\dagger a_3^\dagger\Omega \sim a_2^\dagger\Omega \otimes a_3^\dagger\Omega \tag{5.4}$$

we obtain[4]

$$T_1 a_2^\dagger a_3^\dagger\Omega = a_2^\dagger R_{12} a_3^\dagger R_{13}\Omega \sim a_2^\dagger R_{12}\Omega \otimes a_3^\dagger R_{13}\Omega = (T_1 \otimes T_1)\big(a_2^\dagger\Omega \otimes a_3^\dagger\Omega\big). \tag{5.5}$$

Thus, we are naturally led to the coproduct formula

$$\Delta(T) = T \otimes T \tag{5.6}$$

which is the correct formula for $\mathcal{U}_R$.

Remark 10. Note also that, due to the finite number of a operators in the states of $\mathcal{V}_m$, the vertex operators truncate at level m, and become polynomials in a and $a^\dagger$ in these representations.

6. Examples

We give two examples here: one associated with an additive spectral parameter, and the other with a multiplicative spectral parameter.

6.1. The nonlinear Schrödinger equation

The NLS equation in 1 + 1 dimensions has been widely studied. We look at it in the QISM approach (for a review, see, for example, [5] and references therein).

It has already been shown [2, 4] that all the information on the hierarchy associated with the NLS equation can be reconstructed starting from the algebra $\mathcal{A}_R$, where R is the R-matrix of the Yangian $Y(N)$ based on $gl(N)$:

$$R(k) = \frac{1}{k + \text{i}g}(k\mathbb{I}_N \otimes \mathbb{I}_N + \text{i}gP_{12}) \qquad P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}. \tag{6.1}$$

This R-matrix obey an additive Yang–Baxter equation

$$R_{12}(k_1 - k_2)R_{13}(k_1 - k_3)R_{23}(k_2 - k_3) = R_{23}(k_2 - k_3)R_{13}(k_1 - k_3)R_{12}(k_1 - k_2) \tag{6.2}$$

and one shows, using $P^2 = \mathbb{I}$, that $R_{12}(k)R_{21}(-k) = \mathbb{I}$. Thus, the properties stated above apply.

[4] Care must be taken that the indices 1, 2 and 3 refer to the auxiliary spaces while the tensor product refers to $\mathcal{A}_R$.

In fact, it is well known that the canonical field Φ obeying the (quantum) NLS equation

$$\left(\mathrm{i}\partial_t + \partial_x^2\right)\Phi(x,t) = 2g : \Phi(x,t)\bar{\Phi}(x,t)\Phi(x,t) : \qquad \text{with} \quad \Phi(x,t) = \begin{pmatrix} \varphi_1(x,t) \\ \vdots \\ \varphi_n(x,t) \end{pmatrix}$$

can be reconstructed from $\mathcal{A}_R$ [4]. The Hamiltonian is then exactly $H^{(2)}$, and the Yangian $Y(N)$ is a symmetry of the hierarchy [2, 3]. The operators $a^\dagger$ correspond to asymptotic states in the Fock space $\mathcal{F}$.

The generators Q_0^a and Q_1^a of $Y(N)$ in its Drinfeld presentation are built in terms of $\mathcal{A}_R$ in [2] (see also [3] for the gl_2 case). The present approach is an alternative construction of $Y(N)$ in the FRT presentation. It has the advantage of giving an explicit construction for all the generators of the Yangian, and also of giving the action of these generators (i.e. of the integrals of motion) on the a and $a^\dagger$ operators (i.e. the asymptotic states of the system).

6.2. The quantum group $\mathcal{U}_q(\widehat{gl_2})$

Here, we take the evaluated R-matrix of the centreless affine gl_2 quantum algebra. Following the usual notation, the spectral parameter is denoted as z. The R-matrix reads

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q(1-z^2)}{1-q^2z^2} & \dfrac{z(1-q^2)}{1-q^2z^2} & 0 \\ 0 & \dfrac{z(1-q^2)}{1-q^2z^2} & \dfrac{q(1-z^2)}{1-q^2z^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.3}$$

It is defined here up to a normalization factor ρ such that the unitarity condition $R_{12}(z_1/z_2)R_{21}(z_2/z_1) = 1$ is preserved, i.e.

$$\rho(z)\rho\left(\frac{1}{z}\right) = 1. \tag{6.4}$$

The R-matrix obeys a multiplicative Yang–Baxter equation

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2) \tag{6.5}$$

and once again one can apply the above properties. Note however that we are forced to take a vanishing central charge, so that the algebra $\mathcal{U}_q(\widehat{gl_2})$ is defined by the relation

$$R_{12}(z_1/z_2)T_1(z_1)T_2(z_2) = T_2(z_2)T_1(z_1)R_{12}(z_1/z_2). \tag{6.6}$$

The Hamiltonian $H^{(2)}$ should correspond to the Hamiltonian of the sine-Gordon model.

6.3. The elliptic quantum group $\mathcal{A}_{q,p}(\widehat{gl_2})$

The elliptic quantum group $\mathcal{A}_{q,p}(\widehat{gl_2})_c$ has defining relations

$$R_{12}(z_1/z_2; q, p)T_1(z_1)T_2(z_2) = T_2(z_2)T_1(z_1)R_{12}^*(z_1/z_2; q, p) \tag{6.7}$$

where $R_{12}^*(z; q, p) = R_{12}(z; q, pq^{-2c})$. Note that R_{12} obeys the unitarity condition. Thus, in the centreless case, one has $R^* = R$, and the above procedure can be applied. One starts with the evaluated R-matrix of $\mathcal{A}_{q,p}(\widehat{gl_2})_{c=0}$ and constructs the corresponding ZF algebra.

In this way, one obtains a well-bred vertex operator that realizes $\mathcal{A}_{q,p}(\widehat{gl_2})_{c=0}$, and this latter algebra is a symmetry of the hierarchy associated with the ZF algebra. In particular, the Hamiltonian $H^{(2)}$ should be related to the XYZ model and, in this framework, we naturally obtain $\mathcal{A}_{q,p}(\widehat{gl_2})_{c=0}$ as a symmetry of this model.

7. Conclusion and perspectives

Starting with any R-matrix with spectral parameters, obeying the Yang–Baxter equation and a unitarity condition, we have constructed the corresponding quantum group $\mathcal{U}_R$ in terms of a deformed oscillator algebra $\mathcal{A}_R$. The realization we present is an infinite series, the expansion being given in the number of creation operators. Up to a normalization constant, the construction is unique. These 'well-bred vertex operators' act naturally on $\mathcal{A}_R$. As a consequence, they are integrals of motion of the integrable hierarchy naturally associated with $\mathcal{A}_R$.

Taking as an example the R-matrix of $Y(N)$, the Yangian based on $gl(N)$, using this construction we recover the NLS equation and its $Y(N)$ symmetry. It is thus very natural to believe that the other integrable systems known in the literature can be treated with the present approach.

Of course, a comparison has to be made between the vertex operators constructed in this paper, and the vertex operators of quantum affine algebras known in the literature (e.g. [6]). Note, however, that our construction can be done for *any* infinite quantum group, provided its evaluated R-matrix obeys the unitarity condition.

As a generalization, it is natural to ask whether such an approach can be extended to the case of (elliptic) quantum groups with a non-vanishing central charge; this seems to be very much the case [7]. If such a generalization can be done, it would then be possible to look at (off-shell) correlation functions for the underlying integrable systems. Moreover, this could give a pertinent insight in the research of vertex operators, as they are looked for when starting with the canonical fields of the integrable system [8].

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